ON THE THEORY OF DIFFRACTION OF SHOCK WAVES

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The purpose of the paper is to study the diffraction of a shock wave by a small wedge-like deflection of an otherwise straight wall, perpendicular to the shock front. This problem has also been treated in [1,2,3,4].

Lighthill [1] reduced the problem to a boundary-value problem of Riemann-Hilbert type and solved it by the method of trial and error. A second solution is given below. It is also shown that the solution is not unique because solutions of problems with discontinuous coefficients depend on special conditions which are imposed on the solutions at the points of discontinuity of the prescribed boundary values [5,6]. The choice of a particular class of solutions does not follow from the differential equations and the boundary conditions alone, but requires additional determination.

The method of solution of the boundary-value problem applied here makes possible solutions for more general physical conditions. For instance, solutions for the gas motion can be found when the wall deflects as a result of the oncoming shock wave and also in presence of unsteady disturbances ahead of the shock front which are generated by wall motion.

1. Problem setting. Let the front of the plane shock wave move with velocity V_0 along a smooth wall for $t < 0$, and, at the instant $t = 0$, let it meet a small wedge-like break in the wall of angle $\pm \alpha$, Fig. 1. The medium in front of the shock wave is at rest and is characterized by its density ρ_0, its pressure p_0, and its speed of sound a_0.

The drift flow behind the shock wave is disturbed and the flow field in the disturbed region, bounded by the shock wave, its Mach reflection and the wall, will generally be rotational. Outside this disturbed region, the flow conditions behind the shock wave are constant and are obtainable from relationships across straight shock waves. Let us denote the speed of sound, the flow velocity, the density and the pressure behind the undisturbed original shock wave by a, V_1, R, P, respectively. Depending on the strength of the shock wave, this drift flow can be subsonic or supersonic as is shown in Figs. 1a and 1b.

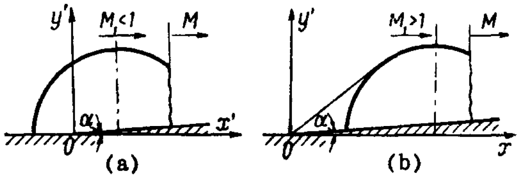

Fig. 1.

Let us describe the disturbed flow field in a coordinate system moving with the drift velocity behind the original shock wave:

$$u = u_1(x', y', t) + \ldots, \qquad v = v_1(x', y', t) + \ldots \tag{1.1}$$
$$\rho = R + \rho_1(x', y', t) + \ldots, \qquad p = P + p_1(x', y', t) + \ldots$$

Furthermore, because of the absence of a length characterizing the problem, the flow properties will be homogeneous functions of order zero of their arguments. Let us introduce nondimensional variables:

$$u_1(x', y', t) = V_1\bar{u}_1(x, y), \qquad p_1(x', y', t) = aV_1R\bar{p}_1(x, y) \qquad \left(x = \frac{x'}{at}\right) \tag{1.2}$$
$$v_1(x', y', t) = V_1\bar{v}_1(x, y), \qquad \rho_1(x', y', t) = R\bar{\rho}_1(x, y) \qquad \left(y = \frac{y'}{at}\right)$$

Hereafter, we shall drop the bar designating the dimensionless variables.

Let us substitute the expressions (1.1) into the basic system of equations, which govern two-dimensional unsteady flows of an ideal gas, and into the boundary conditions at the wall and at the shock wave. The equations for the perturbations are

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\partial p}{\partial x}, \qquad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{\partial p}{\partial y}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} \tag{1.3}$$

At the wall

$$v = \pm a \qquad \text{for } y = 0 \tag{1.4}$$

When the equation of the disturbed shock front is represented in the form

$$x = k + \psi(y) + \ldots \left(k = \frac{V_0 - V_1}{a} < 1\right) \tag{1.5}$$

the relations across the shock wave will take the following form, for $x = k$:

$$M_1 (k + M_1)\, u - k\,(k + M_1)\, \rho = M_1 (\psi - y\psi_y), \quad 2kM_1 u - M_1 p - k^2 \rho = 0$$

$$v = -\psi_y, \quad \frac{\gamma}{\gamma - 1} M_1 p - \frac{1}{\gamma - 1} \rho - kM_1 u = (\psi - y\psi_y), \quad \left(M_1 = \frac{V_1}{a},\ \gamma = 1.4\right) \qquad (1.6)$$

At the Mach-reflected wave front, the perturbations vanish when the drift flow behind the original shock is subsonic. In case of supersonic drift flow, a region of known Prandtl-Meyer expansion or Ackeret compression will be adjacent to the disturbed region along part of the curved Mach reflection, Fig. 1b.

2. Formulation of the boundary-value problem for the function p. Elimination of u and v from the system (1.3) leads to

$$(1 - x^2) \frac{\partial^2 p}{\partial x^2} - 2xy \frac{\partial^2 p}{\partial x\, \partial y} + (1 - y^2) \frac{\partial^2 p}{\partial y^2} - 2\left(x \frac{\partial p}{\partial x} + y \frac{\partial p}{\partial y}\right) = 0 \qquad (2.1)$$

The shock relationships (1.6) can be expressed in terms of the single function p, utilizing (1.3):

$$\left(\frac{2k^2M^2 + M^2 + 1}{2k\,(1 - k^2)\, M^2}\, y - \frac{1}{2ky}\right) \frac{\partial p}{\partial y} - \frac{\partial p}{\partial x} = 0 \qquad \text{for } x = k \qquad \left(M = \frac{V_0}{a_0}\right) \qquad (2.2)$$

At the wall

$$\frac{\partial p}{\partial y} = 0 \qquad \text{for } y = 0 \qquad (2.3)$$

Furthermore, at the corner of the wall, 0, (Fig. 1a), the following condition must be satisfied for the subsonic case [1]:

$$\lim \int_{\Delta x} \frac{\partial p}{\partial y}\, dx = \mp \alpha M_1 \qquad \text{for } \Delta x \to 0 \qquad (2.4)$$

At the reflected Mach wave the condition

$$\frac{\partial p}{\partial s} = 0 \qquad (2.5)$$

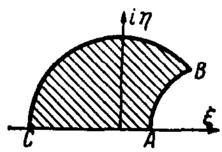

Fig. 2.

must hold, where s represents the direction tangential to the Mach wave.

In the supersonic case (Fig. 1b), the following condition holds at the point of contact between the straight shock or expansion front issuing from 0 and the reflected Mach wave:

$$\lim \int_{\Delta s} \frac{\partial p}{\partial s}\, ds = \frac{\alpha M_1}{\sqrt{M_1^2 - 1}} \qquad \text{for } \Delta s \to 0 \qquad (2.6)$$

The problem is solved when the function p, satisfying Equation (2.1) and all the differential and integral conditions of this paragraph, is determined.

3. Transformation to a boundary-value problem in the theory of analytic functions. By means of the transformation

$$r = \frac{2\varepsilon}{1+\varepsilon^2}, \qquad \theta = \tan^{-1}\frac{y}{x} \tag{3.1}$$

Equation (2.1) can be reduced to Laplace equation in polar coordinates. Let the original physical plane correspond to the plane of the complex variable $z = r \exp(i\theta)$. Then the region of the disturbed flow in the plane $\zeta = \epsilon \exp(i\theta)$ has the shape represented in Fig. 2. Let n denote the exterior normal and s the tangent along the positive direction of the curve AB. The shock-wave relations (2.2) can then be expressed in terms of the normal and tangential derivatives of p along AB:

$$\left(\frac{M^2+1}{2M^2}\tan\theta - \frac{3(M^2-1)}{M^2+5}\cot\theta\right)\frac{\partial p}{\partial s} - \frac{\sqrt{\cos^2\theta - k^2}}{\cos\theta}\frac{\partial p}{\partial n} = 0 \tag{3.2}$$

Let us map the interior of the curvilinear triangle in the ζ plane (Fig. 2) onto the upper half-plane of the complex variable z_1 by means of the conformal representations

$$\zeta_1 = \zeta_B\left(i - \frac{2k_1}{\zeta - \zeta_B}\right), \qquad z_1 = \frac{1}{2}(\zeta_1^2 + \zeta_1^{-2}) \tag{3.3}$$

Here $k_1 = \sqrt{1-k^2}$. The boundary condition at the shock wave, (3.2), which now must be fulfilled on the segment of the real axis $1 < x_1 < \infty$, takes on the form

$$\sqrt{x_1 - 1}\,\frac{\partial p}{\partial y_1} + (Ax_1 - B)\frac{\partial p}{\partial x_1} = 0 \qquad \left(A = \frac{1}{2\sqrt{2}kM^2},\ B = \frac{2M^2+1}{2\sqrt{2}kM^2}\right) \tag{3.4}$$

In the subsonic case, the boundary conditions on the rest of the real axis are

$$\frac{\partial p}{\partial y_1} = \frac{\mp\alpha M_1}{\sqrt{1-M_1^2}}\,\delta(x_1 + x_0) \qquad \text{for } -1 < x_1 < 1 \text{ (on wall)} \tag{3.5}$$

$$\frac{\partial p}{\partial x_1} = 0 \qquad \text{for } -\infty < x_1 < -1 \text{ (on reflected Mach wave)} \tag{3.6}$$

In the supersonic case

$$\frac{\partial p}{\partial y_1} = 0 \qquad \text{for } -1 < x_1 < 1 \tag{3.7}$$

$$\frac{\partial p}{\partial x_1} = \frac{\pm\alpha M_1}{\sqrt{M_1^2 - 1}}\,\delta(x_1 + x_0) \qquad \text{for } -\infty < x_1 < -1 \tag{3.8}$$

Here $\delta(x)$ is the delta function.

4. The solution of the boundary-value problem with discontinuous coefficients. Let us introduce the function

$$P^+(z_1) = \frac{\partial p}{\partial y_1} + i\,\frac{\partial p}{\partial x_1}$$

which is analytic in the upper half-plane and which on the real axis satisfies the condition

$$a(x_1)\frac{\partial p}{\partial y_1} + b(x_1)\frac{\partial p}{\partial x_1} = c(x_1) \tag{4.1}$$

The coefficients in (4.1) are discontinuous:

$$\begin{array}{lll}
a = 0, \quad b = 1 & \text{for } -\infty < x_1 < -1 & \\
a = 1, \quad b = 0 & \text{for } -1 < x_1 < 1 & \\
a(x_1) = \sqrt{x_1 - 1}, \quad b(x_1) = Ax_1 - B & \text{for } 1 < x_1 < \infty & \\
c(x_1) = h\delta(x_1 + x_0) & \text{for } M_1 < 1 & \left(h = \dfrac{\mp \alpha M_1}{\sqrt{1 - M_1^2}}\right) \\
c(x_1) = -ih\delta(x_1 + x_0) & \text{for } M_1 > 1 &
\end{array}$$

so that this boundary-value problem is of Riemann-Hilbert type, with discontinuous coefficients. The solution is not unique, depending upon the nature of the assignable singularities at the points of discontinuity of the coefficients in (4.1). One should seek the solution among the class of functions which are either bounded or have integrable singularities at $x_1 = \pm 1$. Within this class, there are four linearly independent solutions since the solution may at these points be (a) bounded at both points, (b) integrable at both points, i.e. at both points the integral

$$\int P^+(z_1)\,dz_1 \tag{4.2}$$

exists, or (c) bounded at one of the points and integrable at the other, and vice versa.

The lack of uniqueness of the solutions can be seen from the manner in which the boundary-value problem with discontinuous coefficients can be reduced to the boundary-value problem with continuous coefficients, for which the solution is unique, specifically by means of the change in the unknown function

$$P^+(z_1) = \omega(z_1)\,P_1^+(z_1) \tag{4.3}$$

Within the previously specified class of solutions the function $\omega(z_1)$ can be represented by any one of the four possibilities

$$\omega^{(1)}(z_1) = \sqrt{z_1^2 - 1}, \qquad \omega^{(2)}(z_1) = \frac{1}{\sqrt{z_1^2 - 1}}$$
$$\omega^{(3)}(z_1) = \sqrt{\frac{z_1 + 1}{z_1 - 1}}, \qquad \omega^{(4)}(z_1) = \sqrt{\frac{z_1 - 1}{z_1 + 1}} \tag{4.4}$$

Substituting (4.3) into (4.1), we obtain the boundary conditions for the problem with continuous coefficients

$$a_1(x_1)\frac{\partial p_1}{\partial y_1} + b_1(x_1)\frac{\partial p_1}{\partial x_1} = c_1(x_1) \tag{4.5}$$

Here

$$a_1 = 0 \quad \text{for } -\infty < x_1 < 1, \qquad a_1(x_1) = \sqrt{x_1 - 1} \quad \text{for } 1 < x_1 < \infty$$
$$b_1(x_1) = Ax_1 - B, \qquad c(x_1) = -\frac{h(Ax_1 - B)}{\omega(x_1)}\delta(x_1 + x_0) \quad \text{for } -\infty < x_1 < \infty$$

The Riemann-Hilbert problem for a circle or a half-plane can be reduced to the problem of Riemann when the desired analytic function is suitably continued by a piecewise analytic function over the full complex plane [5,6]. The corresponding Riemann problem has the form

$$P_1^+(x_1) = G(x_1)P_1^-(x_1) + g(x_1) \tag{4.6}$$

Here

$$G(x_1) = -\frac{a+ib}{a-ib} = \begin{cases} 1 & \text{for } -\infty < x_1 < 1 \\ \dfrac{Ax_1 - B - i\sqrt{x_1 - 1}}{Ax_1 - B + i\sqrt{x_1 - 1}} & \text{for } 1 < x_1 < \infty \end{cases}$$
$$g(x_1) = \frac{2c_1}{a_1 - ib_1} = \frac{2h}{\omega(x_1)}\delta(x + x_0) \qquad \text{for } -\infty < x_1 < \infty$$

The solution of the modified problem will be given by the function $P_1^+(z)$ which satisfies the "continuation" condition

$$P_1^+(z_1) = \overline{P_1^-(\bar{z}_1)} \tag{4.7}$$

The function $G(x_1)$ can be represented as

$$G(x_1) = \frac{\Phi^+(x_1)}{\Phi^-(x_1)} = \frac{Ax_1 - B - i\sqrt{x_1 - 1}}{Ax_1 - B + i\sqrt{x_1 - 1}} \tag{4.8}$$

on the complete real axis. In this form, it is understood that when one proceeds along the real axis, different branches of the multivalued function $\sqrt{x_1} - 1$ are reached as one passes through the branchpoint $x_1 = 1$; the function $\sqrt{x_1} - 1$ acquires the factor $+ i$ in the denominator, and $- i$ in the numerator. This also follows from the fact that $G(x_1)$ can be represented as a ratio of functions $\Phi^+(x_1)$ and $\Phi^-(x_1)$, which are the boundary values of analytic functions determined in the upper and lower half-plane, respectively. The function $G(x_1)$ will not have zeros or poles all along the real axis. Then

$$\Phi^+(z_1) = \frac{1}{Az_1 - B + i\sqrt{z_1 - 1}}, \qquad \Phi^-(z_1) = \frac{1}{Az_1 - B - i\sqrt{z_1 - 1}} \tag{4.9}$$

Rewriting the boundary condition (4.6) in the form

$$\frac{P_1^+(x_1)}{\Phi^+(x_1)} = \frac{P_1^-(x_1)}{\Phi^-(x_1)} + \frac{g(x_1)}{\Phi^+(x_1)}$$

we obtain

$$P_1^+(z_1) = \Phi^+(z_1)\,[\Psi^+(z_1) + Q_m(z_1)], \qquad P_1^-(z_1) = \Phi^-(z_1)\,[\Psi^-(z_1) + Q_m(z_1)] \tag{4.10}$$

where

$$\Psi(z_1) = \frac{h}{\pi i}\int\limits_{-\infty}^{\infty} \frac{\delta(\tau + \tau_0)}{\omega(\tau)\,\Phi^+(\tau)}\,\frac{d\tau}{\tau - z_1} = \frac{h}{\pi i}\,\frac{1}{\omega(x_0)\,\Phi^+(x_0)\,(x_0 - z_1)} \tag{4.11}$$

Here $Q_m(z)$ represents a polynomial of degree m with arbitrary complex coefficients. The choice of the integer m depends on the choice of the function $\omega(z_1)$ and on the behavior of the solution at infinity. Thus, the general solution of the boundary-value problem (4.1) becomes

$$P^+(z_1) = \omega(z_1)\,\Phi^+(z_1)\,[\Psi^+(z_1) + Q_m(z_1)] \tag{4.12}$$

In order to satisfy the "continuation" condition (4.7) it is sufficient to make use of the fact that the coefficients of the polynomial $Q_m(z_1)$ are real. Furthermore, we shall set $m = 0$ for all functions $\omega(z_1)$, i.e. $Q_0 = C_0$, because $p^+(z_1)$ would otherwise possess zeros at a series of points of the upper half-plane, which is inadmissible on physical grounds. The constant C_0 must be determined from the integral condition

$$\int\limits_0^{k_1} \frac{1}{y}\,\frac{\partial p}{\partial y}\,dy = \frac{\mp\alpha(M^2 + 5)}{3(M^2 - 1)} \tag{4.13}$$

which follows easily from the relations (1.6) on the shock wave.

It is interesting to note that the singularity of the pressure at $x = x_0$ appears as a consequence of the linearization and yet it correctly describes the flow pattern. Figure 1 of [4] exhibits experimental results, which show that in the case of subsonic flow behind the shock wave the increment in the static pressure at the corner of the wall, though finite, dominates the field in magnitude.

5. Pressure distribution on the wall and the choice of the solution. In order to choose the desired solution, let us study the pressure distributions on the wall. The limiting values of (4.12) on the real axis are obtained from Sokhotski's formula. For the case of subsonic flow behind the shock wave, we have for $-1 < x_1 < +1$

$$\begin{aligned} \frac{\partial p}{\partial y_1} &= \frac{\mp \alpha M_1}{\sqrt{1 - M_1^2}} \delta(x_1 + x_0) \\ \frac{\partial p}{\partial x_1} &= \frac{\omega(x_1)}{Ax_1 - B - \sqrt{1 - x_1}} \left[\frac{\pm \alpha M_1 (Ax_0 - B - \sqrt{1 - x_0})}{\sqrt{1 - M_1^2}\omega(x_0)} \frac{1}{x_0 - x_1} + C_0 \right] \end{aligned} \tag{5.1}$$

Let us write the x-derivatives of the pressure for all the functions (4.4) in the original similarity variables

$$\frac{\partial p^{(1)}}{\partial x} = (k - x)^2 \sqrt{1 - x^2}\varphi_1(x), \qquad \frac{\partial p^{(2)}}{\partial x} = \frac{1}{\sqrt{1 - x^2}} \varphi_2(x)$$

$$\frac{\partial p^{(3)}}{\partial x} = \sqrt{1 - x^2}\varphi_3(x), \qquad \frac{\partial p^{(4)}}{\partial x} = \frac{(k - x)^2}{\sqrt{1 - x^2}} \varphi_4(x)$$

where $\phi_j(x)$ $(j = 1, \ldots, 4)$ are continuous functions, which become neither zero nor infinite on the interval $-1 < x < k$, with exception of the point $x = -M_1$, where they are infinite to the first order.

The functions $p(x)$ can be expressed in terms of elementary functions. From the four solutions, preference should be given to $p^{(2)}(x)$, which corresponds to the function $\omega^{(2)}(z_1)$, is identical with the solutions of [1,2] and agrees well with the experiment. The detailed distribution of the pressure $p^{(2)}(x)$ and its comparison with experiment is given in [4].

For the cases which correspond to functions $p^{(1)}(x)$ and $p^{(4)}(x)$, the pressure settles to a constant value immediately behind the shock wave at the point of its intersection with the wall, which is contrary to physical sense. The vanishing of $\partial p^{(1)}/\partial x$ and $\partial p^{(3)}/\partial x$ at the point $x = -1$ is associated with zero values of the pressure disturbance as the corner of the wall is approached from the right at $M_1 = 1$.

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